Scale free distributions

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The Great Wave of Kanagawa, Hokusai 1831.

The central question "Why does one so often observes secular waves ?"

The answer relates to scale free distributions.

Scale free distributions are probability distributions lacking the property of a characteristic element which would give the order of magnitude of the other elements.

In other words, they lack a characteristic scale.

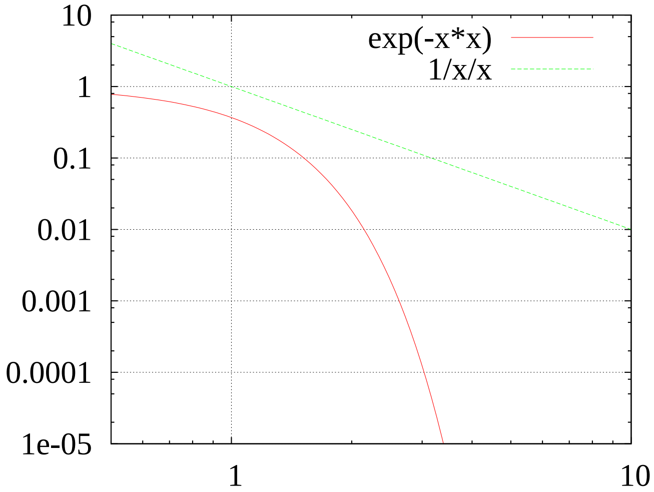

Figure 1: Loglog plot of Gaussian and scale free distributions.

We are familiar with normal distribution such as body size of humans, span of life time, planes or cars sizes, records of athletes etc.

By contrast city sizes in population or area, wealth in capital or revenues, rivers, earthquakes, waves, returns in the stock exchange have scale free distributions: the sizes are distributed on several orders of magnitude.

1 Power law distributions: properties

The most common example of a scale free distribution is the power law:

$$P(x) \sim x^{-\alpha}$$

with positive α , which graph in log-log coordinates is a straight line.

For instance, Gutenberg Wagner law (1944) is written:

$$logN(M) = a - bM$$

where $N(M)$ is the frequency of earthquakes with magnitude M. The magnitude M is defined as the logarithm of the ratio of the amplitude of waves measured by a seismograph to an arbitrary small amplitude. An earthquake that measures 5.0 on the Richter scale has a shaking amplitude 10 times larger than one that measures 4.0, and corresponds to a 31.6 times larger release of energy. Gutenberg Wagner law is then a power law relating the frequency and the energy released during earthquakes.

As contrast to the succession of momenta of bell-shaped distributions, higher momenta of power law distribution diverge, as directly observed from their expression:

$$M(m) = \int_0^\infty x^m x^{-\alpha} dx \quad (1)$$

Momenta are finite for

$$m < \alpha - 1 \quad (2)$$

For $\alpha \leq 1$, even averages are not defined, and for $\alpha \leq 2$ standard deviation are not defined. Of course in practice, when a finite set of empirical data is available, any momentum can be computed; but its value might not give much insight.

Furthermore, when it comes to empirical situations, a power law distribution is not applicable from 0 to ∞. Limits in measurement accuracy and the size of the system e.g. restrict the range of the distribution.

2 Environment

Because of stress propagation, scale invariant distributions are pervasive in environmental risks:

- Physical processes (Earthquakes, waves, tsunamis, avalanches, floods)
- Biological processes (mass extinction in trophic networks, Bak Sneppen)
- Consequences of catastrophes: power grid blackout, consequences of extreme climatic conditions such as blizzard on air, road and train traffic
- and even in rescue (interrupted communications) and recovery (propagation of shortages across production networks)

The above examples illustrate a notion of systemic risk, i.e. large consequences due to the propagation of failures across a network of connected elements.

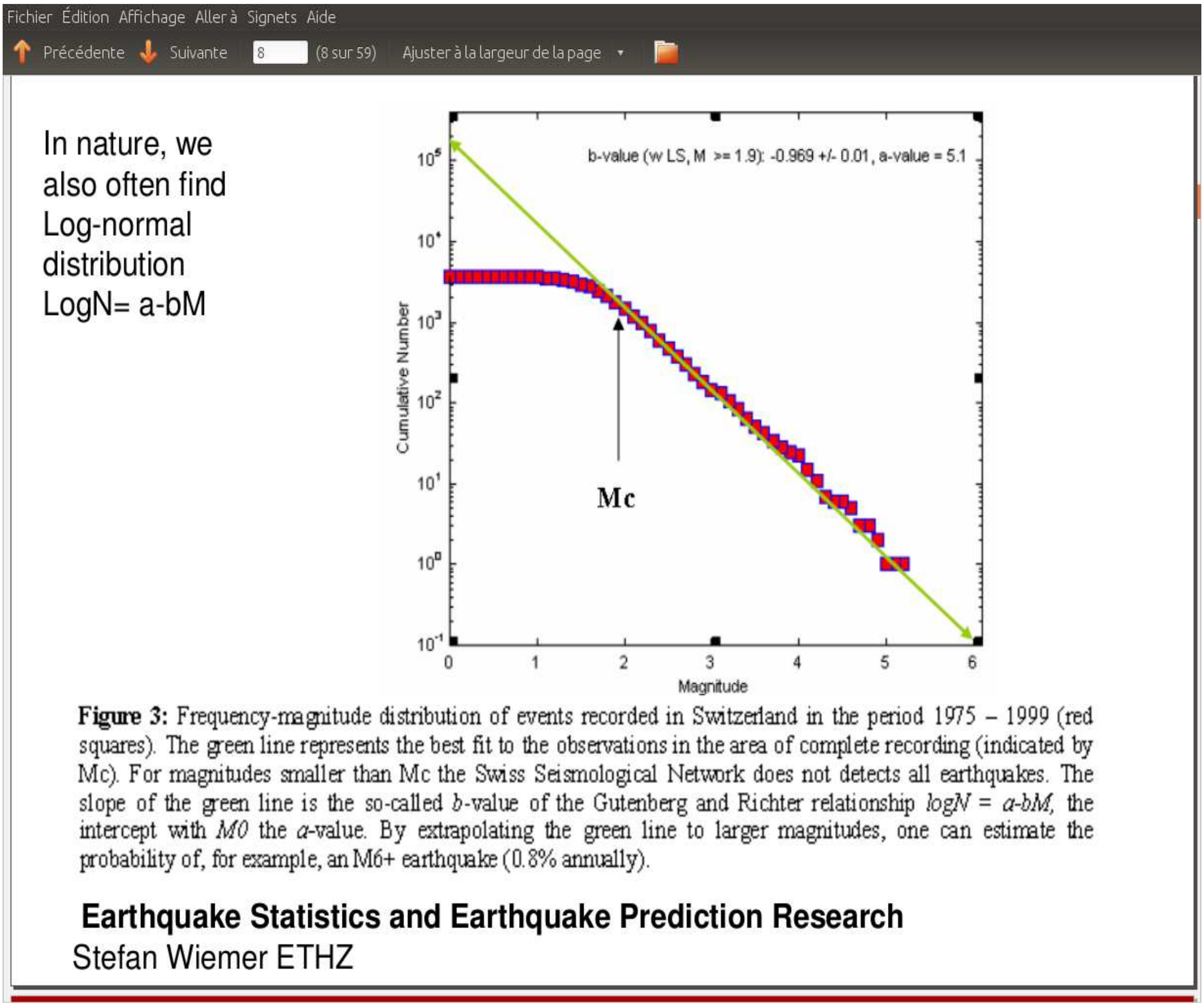

Figure 2: Gutenberg Wagner plot of earthquakes in Switzerland. Note the deviation from the power law at small magnitudes