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# Influence of capital inertia on renewable resource depletion

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**Abstract**

A simple differential equation system describes the dynamics of a renewable resource coupled by a Cobb-Douglas production function to the capital invested in resource harvesting. Capital dynamics is driven by very simple and myopic mechanisms, precluding rational expectations of investors or central planing authorities. This system can model fisheries which operate under open access. Significant resource depletion is shown to follow capital overshoot. We study the influence of biological and economic mechanisms such as the existence of a carrying capacity for the resource, the role of prices and the possibility of a minimum consumption for the labour force on the amplitude of resource depletion. We define a concept of fragility that relates resource depletion to the initial distance of the dynamic variables from equilibrium. An analytic expression of fragility, applicable to differential models of bio-economic systems is derived in this paper. The influence of some institutional interventions is also assessed.

# 1 Introduction

Fisheries provide a good example of the exploitation of a renewable resource and of the problem of stock exhaustion under overexploitation [7]. The standard, but static, approach is based on the concept of a maximum sustainable yield [11, 19]. According to these classical views, [7], economic overfishing is simply the fact that maximum sustainable yield, or rather maximum economic yield, is not achieved because agents keep on increasing their fishing effort as long as any profit can be made. Any factor decreasing costs or increasing revenues, such as technological improvements, could then drive the ecological-economical equilibrium close to resource exhaustion. But the maximum sustainable yield assumption is essentially an equilibrium assumption which does not take into account the fact that the population level of the resource, namely the fish stock, depends upon previous catches, and thus requires a dynamic, rather than static, approach [1, 12]. A number of approaches toward understanding the dynamics of fisheries have been proposed in recent years which depart from the maximum sustainable yield framework, and most of these recent modeling efforts fall into one of two categories: the control approach [7, 6]; or the case history approach for describing a specific place and time sequence [2, 7, 16, 10]. (Of course, some authors do use parameters estimated from real case studies in the control approach, e.g.[4]). However, the idea of control presupposes a central authority able to enforce the rules that optimize the catches, a situation seldom encountered in reality, except in places where access is restricted to a small number of boats. The case history approach often relies upon including such a large number of details about the particular case being studied that both interpretation and generalization of the results to other situations sometimes become difficult. Furthermore, one of the purposes of a case study is to validate a model by adjusting its parameters in order to obtain a best fit to available data, while the approach taken by this paper is more general: our main purpose is to get some insight in the mechanisms responsible for resource depletion and the efficiency of factors that could decrease it. In our view, resource depletion is most often due to inertial factors in the dynamics of exploitation, such as labor and capital. Since these aspects are seldom taken into account in previous modeling studies, one exception being [15, 14] for instance, the emphasis in this paper is on simplification to improve understanding. Dealing with problems characterized by a lot of variability and uncertainties, we are looking for semi-qualitative properties such as the qualitative behavior of the observed dynamical regimes with respect to parameters and their scaling laws.

We consider the case where the resource is harvested under conditions of open access, without a central control. Our main concern is the depletion of the resource in relation to the dynamics of both the natural resource and the amount effort put into harvesting the resource. But rather than supposing that the effort adjusts rapidly to changing circumstances, at a rate proportional to the difference between costs and benefits, as in [7], for example, our assumption is that the industry readjusts with some inertia caused by the capital and labour already engaged in the economic activity. Most bio-economical models taking into account dynamics of capital, labour or prices are within the framework of a perfectly fluid market and economic rationality [7]. As one can imagine, a number of real fisheries are working far from these assumptions [5]. Labor and capital market are not perfectly fluid:

- a fisherman does not instantly become a farmer when his income falls below some reference salary, and fishing gear such as boats are not instantly changed into farming equipment when returns on capital invested in fisheries are too low. Many fisheries are traditional activities, especially in Europe and in developing countries, and their agents don't behave as traders on the stock market: they might not engage in other more profitable activities due to lack of information and capabilities.

- Social pressures acting to preserve a way of life with its related social structures, might also prevent them from doing so (see the discussion on fishing communities in [5]).

- Furthermore, economic rationality supposes that the agents have a perfect knowledge of the resource, including its dynamics. In fisheries, economic agents have only rough estimates of the fish stock and the carrying capacity of marine environment can have very large and unexpected variations (e.g El Niño 1973 effect on anchovy fisheries).

The issue of the irreversibility of investment in resource management has been discussed by a few authors, including [4]... The simplified model that we use in this paper to describe fisheries basic dynamics resembles their basic models. But those previous papers suppose optimal control of exploitation by the investment level for instance while we use an automatic reallocation rule described in section 2.

The present paper is organized in sections describing models of increasing complexity. To simplify the analysis, we have chosen capital as the only economic variable whose dynamics is coupled to that of the resource. In order to derive general results, and facilitate their interpretation, we work with a pair of differential equations which describe the coupled time dynamics of the resource, which is the fish population or its total mass in the case of fisheries, and the capital. The behavior of this simple system of differential equations is easily analysed by linearizing near the equilibria and by doing computer simulations using a standard o.d.e. integrator [9]. We start in section 2 from a very simple reallocation rule such that profits from the fishery, namely catch minus capital depreciation, are shared according to a constant fraction between consumption and capital reinvestment. This automatic reallocation rule corresponds in fact to an extremely myopic behavior where agents only take into account present profits when deciding how to reallocate them and to extreme traditionalism since they don't care about capital and labour market. In real life all these aspects influence the behavior of the agents. But rather than refining traditional economic modeling to better fit insufficient data from some given fishery, we want to understand the consequences of capital and labour inertia. In some sense, rather than supposing that fisheries are perfectly coupled to the rest of the economy, we start from the opposite simplifying assumption, namely that they are completely decoupled. In section 2, the oversimplification of a first model proposed by Roughgarden and Brown [17] allows us to gain some insight in the dynamical behavior of the model and to derive analytical expressions of the relevant quantities. This simple model evolves towards equilibrium, but only after overshooting the equilibrium significantly. Following the overshoot period, the resource is dramatically depleted. Our main focus is the study of the depletion that follows the overshoot. The interest of this first section is mainly heuristic: since the differential system only contains one reduced parameter, the influence of this parameter on depletion is easily demonstrated.

By introducing several refinements in the subsequent sections, 3 to 5, we are able to use the methods and the concepts defined in section 2 to investigate how the dynamics described by each model depend on the particular processes that are included in it as well as how they change with the parameter values. We will consider one-by-one the influence of a natural carrying capacity for the resource (section 3), the effect of harvest prices on the market (section 4), and the effect of processes that keep consumption by the labor force always above some minimum level (section 5). We investigate to what degree these economic and biological mechanisms attenuate the initial resource depletion observed in the simplest model. The role that some of the market control measures commonly employed by planning authorities, such as governments, might play in stabilizing the market will be also discussed. In each case we concentrate on the implication of this series of models for fisheries with open access. However, one aim of the present study is to obtain results of a general nature that apply to other renewable natural resources such as forests or clean water.

# 2 The simplest model: coupling resource and capital dynamics

## 2.1 Model description

The diagram of our simplest model is presented in figure 1.

Resources, $N$, are renewed at a constant proportional rate $r$. $N$ can be interpreted as a fish population, or better as the fish total mass. Resources are

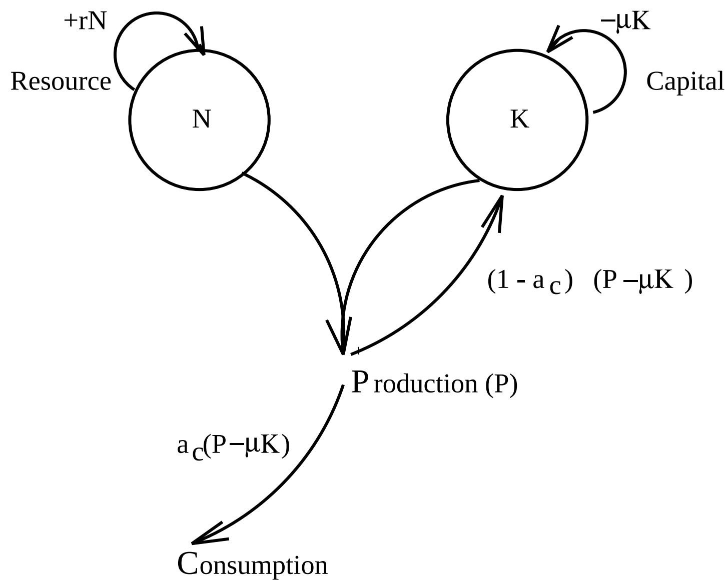


Figure 1: Diagram of the different processes involved in the dynamical model.

depleted because of harvesting by the fishermen. In this model the harvest obeys a generalized Cobb-Douglas expression:

$$P = \sqrt{KL}N, \tag{1}$$

where $P$, the production function, has decreasing returns in terms of the labor force, $L$, and the invested capital, $K$. This simple Cobb-Douglas expression, with half powers on $K$ and $L$ and a unitary power on $N$, was chosen to allow simple algebraic computations rather than to be realistic. Still, proportionality of production to $N$ is a common assumption of many fisheries models ([7] and [22] for instance). It corresponds to the hypothesis of a constant catch per unit effort. In the present model the effort is represented by the square root term in $K$ and $L$. This term gives a constant return to scale and perfect substitution between $K$ and $L$. We discuss in section 2.4 a possible generalization of equation 1 and some consequences specific to the present model.

We further distinguish between $P_k$, the size of the harvest in monetary units, and $P_n$, the size in number of fish, by multiplying $P$ by coefficients $a_k$ and $a_n$ for, respectively, the size of the harvest in monetary units and in number of fish. $a_k/a_n$ can be thought of as a price.

The labor force is assumed to be constant in time. Let $\mu$ be the capital depreciation rate, i.e. the inverse mean life time of the boats represented by the capital, $K$. $(P_k - \mu K)$ can be thought of as the profit from the fishery, coming from the production in monetary units after compensation for capital depreciation. Our main assumption in this section is that a constant fraction of the profit, $a_c$, is spent on consumption. The rate of consumption, $C$, is assumed to be

$$C = a_c(P_k - \mu K). \tag{2}$$

The profits from the harvest are then either consumed by the fishermen or reinvested into new capital in order to increase future yields. In some sense, investment and consumption do not correspond to any optimization decided by fishermen upon some economic rationality, but simply to what they can invest or consume. This fixed reallocation ratio, independant of the size of the profits, differs from the usual assumptions of equilibrium economics, in which capital and labor instantly quit the activity when profits plumet. A similar fixed reallocation ratio was observed during the twentieth century by A. Bowley in England and P. Douglas in the United States and is known in evolutionary economics as Bowley's law ([18]). Bowley's law is taken here as a simple first approximation to test the influence of capital inertia. The following set of equations describe the coupled resource and capital dynamics:

$$\dot{N} = rN - a_n P, \tag{3}$$

$$\dot{K} = (1 - a_c)(P_k - \mu K). \tag{4}$$

In this simple form the model is reminiscent of a Volterra-Lotka predator prey model, where the capital plays the role of the predator and the resource that of the prey population. It also resembles some of the two differential equation models early proposed by [20], although Smith's differential equation for capital is based on different assumptions. But the main difference between Smith's approach and the present study is that Smith focuses on equilibrium while we are interested in resource depletion which here appears as a dynamical transient process (see further).

## 2.2 Simulation results

Figure 2 displays the capital and the resource as they vary with time. Solutions are damped oscillations which gradually converge to an equilibrium point. Note that when the initial conditions are not near equilibrium the first oscillation is quite strong: this corresponds to a real depletion of the resource, which would have a dramatic impact on the income level of labour. The present paper focuses on this effect. Of course one might imagine other initial conditions, but there is no reason to suppose that resource exploitation starts anywhere close to a production equilibrium. Instead, for previously unexploited or lightly exploited resources, the resource usually starts at some high level and capital starts from a low value since it has not had sufficient time to accumulate. The large overshoot in capital observed in figure 2, followed by extreme resource depletion, corresponds to the most dramatic scenarios described, for example, in Beyond the Limits (p. 108) [15, 14].

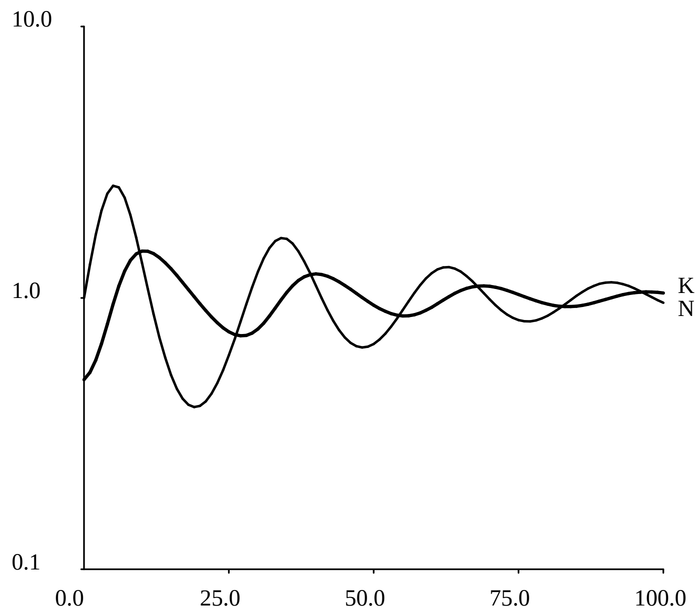


Figure 2: Logarithmic plot of the time variations of the normalized capital, $\tilde{K}$, and the normalized resource, $\tilde{N}$, for the simple model of section 2. Normalized variables and parameters are discussed in section 2.3. Time is scaled in units of $1/r$, and this scaling is the same for all time plots. Initially the resource is at equilibrium ($\tilde{N}(0) = 1$), while the capital is at one half of the equilibrium ($\tilde{K}(0) = 0.5$). Here $\beta = 0.1$. Note the strong resource depletion during the first oscillation, even though initial conditions are not far from equilibrium.

One of the purposes of the present paper is to study the magnitude of this resource depletion under the influence of a number of mechanisms which were supposedly absent from the Limits to Growth model, such as technological improvements, prices, and dynamics of consumption...and compare how much each mechanism reduces the level of resource depletion. Resource depletion is systematically measured, using numerical simulations, as a function of the fraction of initial capital with respect to the final equilibrium capital, with the initial resource being taken to be its equilibrium value (see figures 3, 6 and 14). Resource depletion can also be analytically computed in the neighborhood of the equilibrium point using the linear approximation. This is done in section 2.3.

## 2.3 algebraic approach

This system has a nontrivial equilibrium state, given by $N = N^*$ and $K = K^*$, where

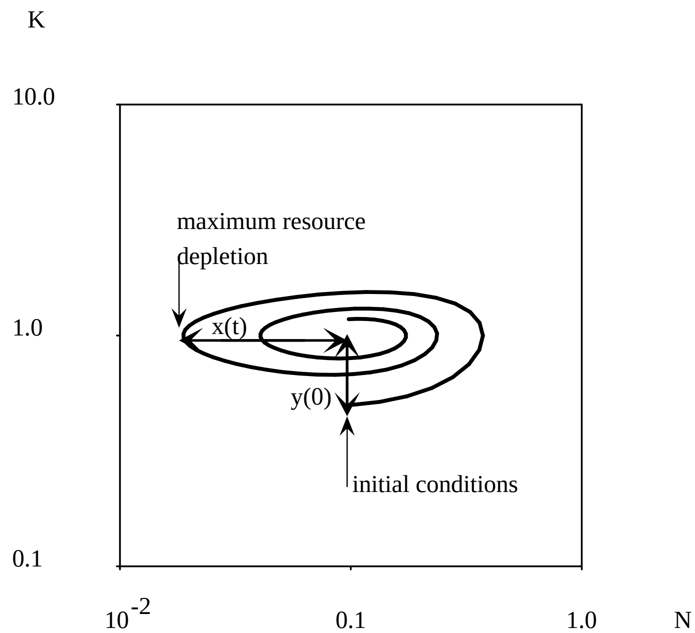


Figure 3: Trajectory in the phase plane showing the initial conditions, with a deviation y(0) of the capital from its equilibrium value, and the point of maximum resource depletion, x(t), used in figures 6 and 14 diagrams.

$$K^* = \left(\frac{r}{a_n}\right)^2 \frac{1}{L}; \quad N^* = \frac{r\mu}{a_n a_k L}. \tag{5}$$

In the rest of this paper the algebra and simulations will be performed with non-dimensionalized systems. Nondimensionalizing the system, by changing variables, reduces the number of parameters in the system and allows us to examine all at once a set of parameters which lead to the same *mathematical* behavior. For example, as we will see, all parameter choices with the same values of $(1 - a_c)\mu/r$ yield the same dynamical behavior, no matter what $r$, $a_c$ and $\mu$ are. In order to interpret results, we will sometimes come back to the original variables and parameters.

We use the equilibrium values of $N$ and $K$, and the rate of resource replenishment $r$, to nondimensionalize the system by letting

$$\tilde{N} = \frac{N}{N^*}; \quad \tilde{K} = \frac{K}{K^*}; \quad \tau = rt \tag{6}$$

This gives the nondimensionalized system:

$$\frac{d\tilde{N}}{d\tau} = (1 - \sqrt{\tilde{K}})\tilde{N} \tag{7}$$

$$\frac{d\tilde{K}}{d\tau} = \beta(\sqrt{\tilde{K}}\tilde{N} - \tilde{K}) \tag{8}$$

with only one parameter remaining:

$$\beta = \frac{(1-a_c)\mu}{r}. \tag{9}$$

$\beta$ gives a measure of the ratio between the rate of loss of capital and the replenishment rate of the resources.

This system has two equilibria at $(\tilde{N}, \tilde{K}) = (0,\ 0)$ and $(1,\ 1)$. The equilibrium around the origin is unstable. Linearizing around the origin, the two equations separate, and $N$ grows exponentially at a rate of 1, while $K$ decays exponentially at a rate of $\beta$.

Linearizing around the point (1,1) gives:

$$\tilde{N} = 1 + x; \quad \tilde{K} = 1 + y \tag{10}$$

$$\frac{dx}{d\tau} = -(1/2)y \tag{11}$$

$$\frac{dy}{d\tau} = \beta(x - y/2), \tag{12}$$

which has eigenvalues $\lambda = \frac{-\beta \pm \sqrt{\beta(\beta-8)}}{4}$. This equilibrium is always stable. If $\beta \geq 8$ this equilibrium is reached without oscillations. If $\beta < 8$ oscillations decay toward the equilibrium. The magnitude of the oscillations of course depends on the initial conditions, being larger the further the initial system is from equilibrium (figures 6 and 14). These oscillations are the dramatic oscillations that are observed by numerical simulations of figure 2 for instance. In the limit of small $\beta$, the oscillation frequency is:

$$\frac{2\pi\sqrt{8\beta}}{4}.r = \pi\sqrt{2(1-a_c)\mu r} \tag{13}$$

where the $r$ factor comes from changing back the time scale from $\tau$ to $t$.

Resource depletion as a function of the distance of the initial conditions from equilibrium can also be estimated analytically, at least when solutions are so close to equilibrium that the linear approximation is valid. Figure 3 shows that we have to relate $x(t)$, the relative resource depletion at time $t$, where $t$ is the time for the system to rotate $3\pi/2$ around equilibrium, to $y(0)$, the initial relative deviation of capital from equilibrium (relative here refers for instance to the ratio of initial capital to capital at equilibrium).

$$x(t) = x(0)exp(\frac{3\pi\lambda_r}{2\lambda_i}), \tag{14}$$

where $\lambda_r$ and $\lambda_i$ are the real and imaginary part of $\lambda$. The ratio of $x(0)$ and $y(0)$ is obtained from equation 13, where the time derivative of $x$ is replaced by $\lambda x$.

$$x(0) = \frac{y(0)}{2\lambda} \tag{15}$$

We can then define the linear fragility $f$ of any bio-economic system with overshoot and depletion by:

$$f = \frac{x(t)}{y(0)} \tag{16}$$

which gives:

$$f = \frac{exp(\frac{3\pi\lambda_r}{2\lambda_i})}{2\lambda}, \tag{17}$$

for system 3-4. High fragility refers to a strong resource depletion following any initial departure for equilibrium. Linear fragility can a priori take any value between 0 and infinity. The variations of $f$ with $\beta$ are represented on figure 4.

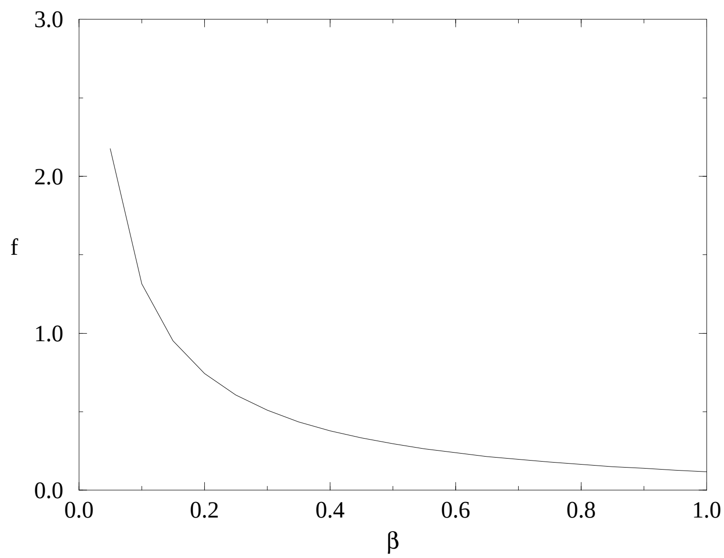


Figure 4: Linear fragility, defined in the text as the ratio between maximum relative resource depletion to initial relative capital distance to equilibrium ( see figure 3) is represented as a function of $\beta$ defined by equation 9. Computation is done according to equation 17. The fragility of the system increases when the renewal rate of the resource is much faster than capital depreciation rate.

## 2.4 Interpretations

Let us come back to the non-dimensionalized system in order to interpret these results. The unstable equilibria are never reached and are of little interest. Let us focus on the stable equilibrium and on the evolution equilibrium.

Since the oscillations around equilibrium are the major features of the dynamics of this model, let us discuss here their condition of existence in the same class of models. As shown by the same linear analysis, if we consider a generalized

Cobb-Douglas function with elasticities u, v and w different from 0.5 or 1 such as:

$$P_k = a_k K^u L^v N^w \tag{18}$$

the real part of $\lambda$ is :

$$\lambda_r = -\frac{\beta(1-u)+w-1}{2}. \tag{19}$$

The stability of the limit point is maintained as long as $\lambda_r$ is negative. For constant catch per unit effort ($w = 1$) this is true when $u$ is smaller than one, which corresponds to decreasing returns in capital. However, instabilities could occur because of decreasing returns with respect to the stock of fish ($w < 1$), when $\beta$ is small. Some observations of [16] on anchovy fisheries, with $w = 0.4$ correspond to this latter case.

The equilibrium resource size $N^*$ increases with the renewal rate of the resource $r$, the capital depreciation rate $\mu$, and decreases with the rate of production $a_k$, the efficiency of the harvesting $a_n$ and the amount of people involved in the harvest, $L$ (equation 5). The fact that the equilibrium capital $K^*$ is inversely proportional to $L$ simply reflects the replacement of capital by labour in the Cobb-Douglas expression 1. $K^*$ also increases with $r$ and decreases with $a_n$.

$\beta$ is small, giving rise to oscillations, whenever resource renewal is fast with respect to capital depreciation, which is probably the case for most fisheries. The resource renewal rate for most species of fish is of the order of a few years, while boats last for a couple of decades at least (major pieces of fishing gear typically last for seven years). The situation for forest exploitation might be the opposite, with $\beta \geq 8$ and no oscillations and a fast decay towards equilibrium.

An important result is that $\beta$ is independent of the production coefficients $a_n$ and $a_k$: $a_k$ can be increased by technological or marketing improvements that decrease the quantity of a natural resource used in a given product. This would not change the oscillatory nature of the solutions. In the case of forest exploitation such an improvement might be using less wood for a given piece of furniture or making use of a larger proportion of each tree; for fisheries it might be practices enabling the industry to use a larger fraction of the catch by marketing surimi or flour for cattle breeding. On the other hand, such practices can improve the situation by increasing the equilibrium resource size, which is a good defense against natural or artificial hazards which could suddenly deplete the resource below the threshold level necessary for reproduction.

# 3 Carrying capacity

A number of renewable resources are limited in size, even in the absence of human exploitation. In presenting the model of the previous section we have assumed that either the resource level was far from this limit or else that the limit did not affect significantly the behavior of the system. Whatever the specific mechanisms involved in the actual limitation of a given resource, which might be food or space limitations or the existence of predators, the concept of a carrying capacity $m$ [13] is easily introduced by a decrease of the renewal rate represented in the dynamics of the resource by replacing equation 3 with:

$$\dot{N} = rN(1 - \frac{N}{m}) - a_n P \tag{20}$$

## 3.1 Algebraic approach

Unless otherwise stated, in the following sections all changes of variables are done with respect to $N^*$ and $K^*$ defined by eq. 5 of section 2, even though those quantities might not be the equilibrium values of the modified model used in that section. $N^*$ and $K^*$ will be further referred to as the *harvesting equilibrium* in the sense that they are the equilibrium values for a model with only one negative term in the resource equation, namely the harvest term.

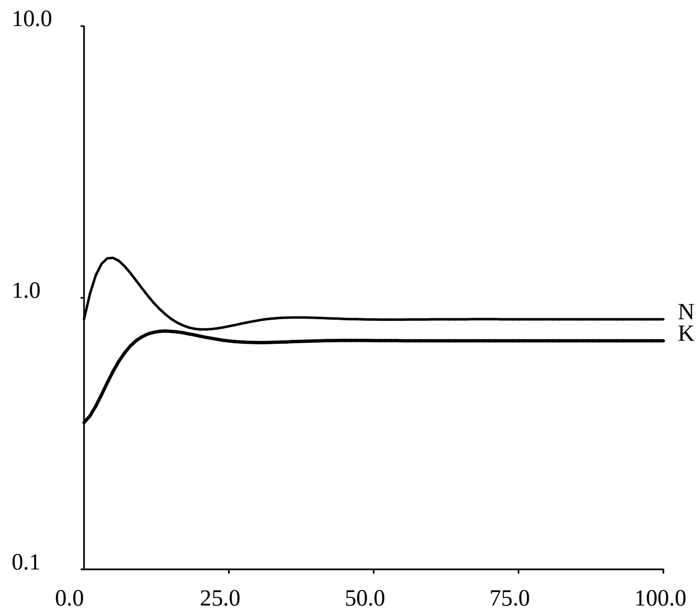


Figure 5: Time plot of the resource level and capital for the model of section 3, which accounts for situations where the resource is of the same order as the carrying capacity. $N$ and $K$ are normalized by $N^*$ and $K^*$ of section 2, respectively. The first upward oscillation of the resource is damped by the presence of the carrying capacity, resulting in a smaller amount of resource depletion. In this figure $\tilde{m} = 5$, $\beta = 0.1$, the initial resource level is taken to be at the equilibrium of $\tilde{N} = 0.83$, and the initial capital at half of its equilibrium value, so that $\tilde{K}(0) = 0.35$.

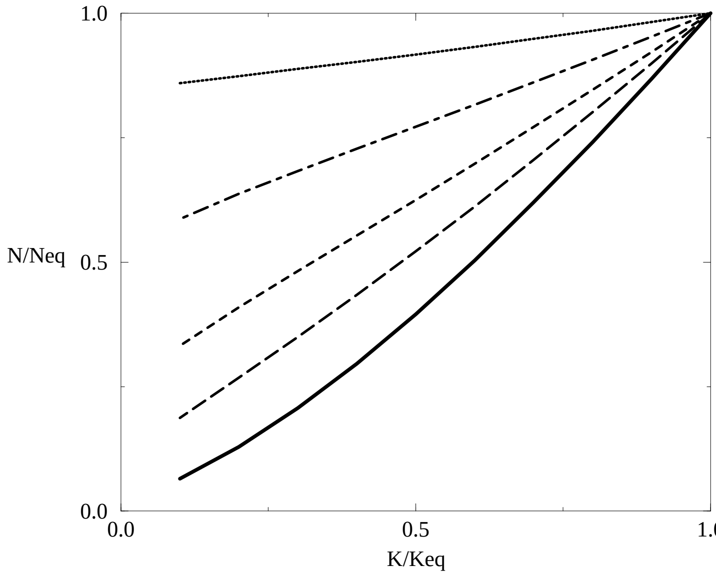


Figure 6: Influence of the carrying capacity on resource depletion. The magnitude of the resource depletion is plotted on the vertical axis as a function of the distance from equilibrium for each of the four values of the ratio between carrying capacity and harvesting equilibrium (40, long dashes; 20, short dashes; 10, dot dash and 5 dotted line) and for the model of section 2, solid line, which would correspond to an infinite carrying capacity. The existence of a carrying capacity reduces resource depletion, even when the carrying capacity is rather high with respect to the harvest equilibrium.

The carrying capacity can be normalized the same way as $N$, *i.e.* with respect to $N^*$ given by eq. 5, giving $\tilde{m} = m/N^*$, and the new nondimensionalized model is:

$$\frac{d\tilde{N}}{d\tau} = (1 - \sqrt{\tilde{K}} - \tilde{N}/\tilde{m})\tilde{N} \tag{21}$$

$$\frac{d\tilde{K}}{d\tau} = \beta(\sqrt{\tilde{K}}\tilde{N} - \tilde{K}). \tag{22}$$

This model has three equilibria at $(\tilde{N}, \tilde{K}) = (0,0)$, $(\tilde{m}, 0)$ and $(\tilde{m}/(1 + \tilde{m}), \tilde{m}^2/(1+\tilde{m})^2)$.

These first two equilibria are unstable.

Near the usual attractor, set $\tilde{N} = \tilde{m}/(1+\tilde{m}) + x$, $\sqrt{\tilde{K}} = \tilde{m}/(1+\tilde{m}) + z$ and linearize to get:

$$\frac{dx}{d\tau} = \frac{-\tilde{m}}{1+\tilde{m}}(\frac{x}{\tilde{m}} + z); \tag{23}$$

$$\frac{dz}{d\tau} = \frac{1}{2}\beta(x - z); \tag{24}$$

The eigenvalues are

$$\lambda = \frac{1}{2}\left[-\beta' \pm \sqrt{(\beta')^2 - 2\beta}\right], \tag{25}$$

where $\beta' = \beta/2 + 1/(1+\tilde{m})$ .

## 3.2 Interpretation

The existence of a carrying capacity reduces the equilibrium value for the resource by a factor of $1 + \frac{N^*}{m}$ relative to the harvesting equilibrium, $N^*$, of section 2. Alternatively, harvesting reduces the level of the natural resource by a factor of $1 + \frac{m}{N^*}$ relative to the carrying capacity.

The absolute value of the real part of $\lambda$, which determines the strength of the attraction towards the equilibrium, is strongly increased when the carrying capacity is of the same order as the harvesting equilibrium $N^*$. It can be observed on the time plot of figure 5 that the carrying capacity limits the upward oscillation of the resource, thus limiting the overshoot and the subsequent backlash. This corresponds to the results shown in figure 14: the resource depletion curve for a carrying capacity of five times the harvesting equilibrium is well above the same curve in the absence of a carrying capacity. In other words, if resources were harvested at a rate such that the harvesting equilibrium were of the same order of magnitude as the carrying capacity then resource depletion would not be a big issue, even for small values of $\beta$.

Using equation 24, the linear fragility of the system is now given by

$$f = exp(\frac{3\pi\lambda_r}{2\lambda_i})(\frac{2|\lambda|}{\beta} + 1). \tag{26}$$

The variations of $f$ with $\tilde{m}$ are represented on figure 7.

The stronger attenuation of the oscillations by the carrying capacity is accompanied by a decrease in the size of the $\beta$ region in which oscillations occur. The following values of $\beta'$ limit the oscillation region:

$$\beta' = 2(1 \pm \sqrt{\frac{\tilde{m}}{1+\tilde{m}}}). \tag{27}$$

For instance, when the carrying capacity m is large compared to the harvesting equilibrium $N^*$, the upper range in $\beta$ of the oscillatory region is decreased to $8 - 4/\tilde{m}$ instead of the limit of 8 which was found for the system without carrying capacity.

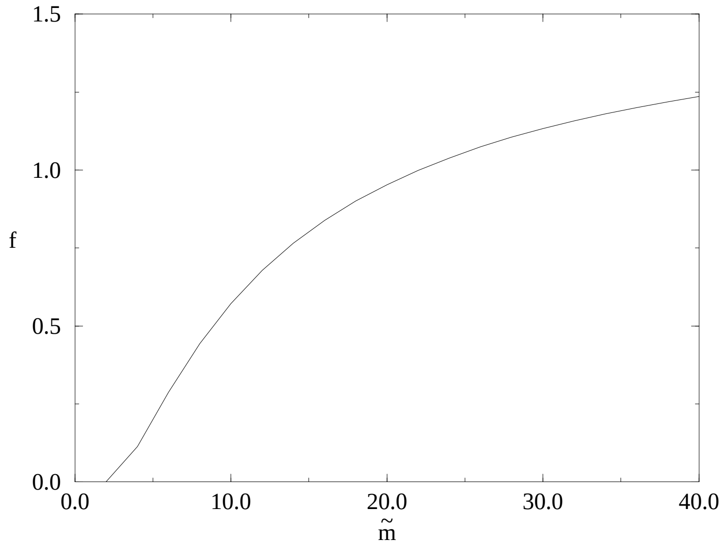


Figure 7: Linear fragility, defined in the text as the ratio between relative resource depletion to initial relative capital distance to equilibrium (see figure 3) is represented as a function of $\tilde{m}$, the ratio of the carrying capacity to the harvesting equilibrium for a $\beta$ coefficient of 0.1. Computation is done according to equation 26. Resource depletion is strongly attenuated by the existence of a carrying capacity of the same order as the harvesting equilibrium.

# 4 Prices

## 4.1 The model with prices

Economists have raised strong objections about the dire outcomes predicted by envionmentalists using systems dynamics models, such as that of the Club of Rome [14]. One of their criticisms concerns the ability of market mechanisms to damp dangerous oscillations [21, 8]. The argument is that when a resource becomes rare its concomitant price increase lowers its exploitation, or favors technologies less wasteful in resource usage.

Let us rewrite the production in monetary units as

$$P_k = a_k(P)P, \tag{28}$$

where

$$P = \sqrt{KL}N, \tag{29}$$

in order to introduce a production in value which depends upon the production in fishes, instead of the previous expression which contained only a constant production coefficient, $a_k$.

The differential equation for the resource, $N$, remains unchanged. (We use equation 3 from section 2 without the carrying capacity term). The only change occurs in the differential equation for the capital:

$$\frac{dK}{dt} = (1 - a_c)(a_k(P)P - \mu K). \tag{30}$$

A simple monetary coefficient function $a_k(P)$ is given by:

$$a_k(P) = \frac{b}{P + c} \tag{31}$$

and the differential equation for the capital is then:

$$\frac{dK}{dt} = (1 - a_c)(\frac{bP}{P + c} - \mu K). \tag{32}$$

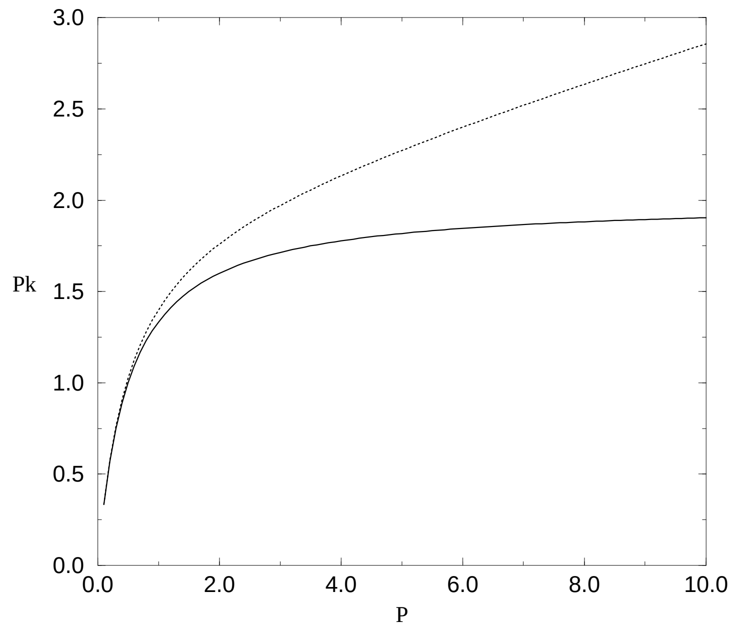


Figure 8: Plot of the production expressed in monetary units, $P_k$, as a function of production, $P$, for the functions used in section 4. The solid line corresponds to "market" prices with coefficients $b = 2$ and $c = 0.5$ of the first model, equation 31. Note the plateau region which gives a constant income at large yield. The dotted line corresponds to the second model with a minimum sustained price $a = 0.1$, with the same values for $b$ and $c$, equation 37.

Figure 8 indicates a simple interpretation of the coefficients $b$ and $c$ in the fraction $a_k$: $b$ is a maximum production in value, obtained when the catch goes to infinity, and $c$ is a characteristic scale on the production $P$. $\frac{b}{ca_n}$ is the maximum price, which is obtained at zero production. We might consider that the above function reflects the situation where there is a strong decrease of prices with production since value production saturates at high production: there remains no incentive for producers to increase the catch in this regime. Under "normal" circumstances, the decrease of $a_k(P)$ with $P$ is probably less strong than in expression 31. Price dynamics under normal economic conditions are probably intermediate between the models of section 2 and 4.

## 4.2 Algebraic analysis

The equilibrium resource level $N_p^*$ is obtained by equating the left-hand side of equation 32 to zero:

$$N_p^* = \frac{\mu a_n L c}{r(b - \mu K^*)} \tag{33}$$

A positive resource equilibrium is only achieved when the maximum production $b$ is larger than the capital depreciation rate during the same period, $\mu K^*$, otherwise the capital keeps on decaying. The equilibrium resource gets bigger and bigger as the maximum yield gets closer to the capital depreciation rate.

Now that both $N_p^*$ and $K^*$ are known the stability analysis can be done around the equilibrium.

The following expression is obtained for the stability parameter $\lambda$:

$$\lambda \sim \frac{-\beta(b'+1) \pm \sqrt{\beta^2(b'+1)^2 - 8\beta b'(b'-1)}}{4b'}, \tag{34}$$

where $b' = \frac{b}{\mu K^*}$. Note that the dynamical behavior only depends upon $\beta$ and $b'$, but not on $c$.

Damping of oscillations, related to the real part of $\lambda$, is increased by prices by a factor

$$1 + 1/b' = 1 + \frac{\mu K^*}{b}. \tag{35}$$

Prices do attenuate the oscillations, and thus resource depletion, but since the fraction should be smaller than one for equilibrium to exist, the damping factor is at most 2.

The region of oscillation is decreased by prices. Oscillations occur when

$$\beta < \frac{8b'(b'-1)}{(b'+1)^2}. \tag{36}$$

In fact expression 31 refers to a situation when demand is large and when prices are adjusted by classical supply demand adjustment mechanisms. This is generally the case for artisanal fisheries of highly valued species such as sole, cod, haddock, shrimps and lobsters. But for some fisheries, such as french industrial fisheries of hake, herring, anchovy, market demand can be low with respect to production. In this context, in order to preserve the economic fishing sector, landing prices are maintained by institutions such as government or producer organizations. We can further complicate the monetary coefficient function $a_k(P)$ by introducing a minimum price $a$ for fish (see figure 8), to model the case of a minimum price maintained by some institution:

$$a_k(P) = \frac{aP + b}{P + c}. \tag{37}$$

A full algebraic analysis of this system, reported in the Appendix, has been done which gives expressions which are difficult to interpret; but we can still predict the two extreme dynamical regimes. When the minimum price is low, the major difference is that the restriction on $b$ no longer holds since a production equilibrium always exists. At low equilibrium production the minimum price can be neglected and the above analysis (beginning of this section) makes it possible to predict the dynamics. In the large production region the standard analysis of section 2 applies with $a_k = a$ as if the price were simply $\frac{a}{a_n}$. Large resource depletion is observed, and the equilibrium resource is maintained at a level inversely proportional to $a$.

### 4.3 Numerical results

Numerical simulations do confirm the role of prices in decreasing the amplitude of oscillations and in reducing resource depletion as long as the equilibrium production is in the region where the prices are decreasing with production, *i.e.* when the demand is large with respect to production (see figure 9 and figure 14). When the equilibrium demand is small compared to production and when the prices are maintained by the government or another institution such as a producers organization, oscillations and resource depletion are of course little reduced (see figure 9).

## 5 A model with a lower bound on consumption

In the previous sections, the consumption was taken to be proportional to the profits, ie $C = a_c(a_k P - \mu K)$. This assumption was not meant to describe consumption patterns when profits are low: consumption would become negative with negative profits. In fact the consumption by itself did not directly play a role in the mathematical model, since the differential equation for the capital contained only the replenishment term which is simply proportional to the profits. A term like this corresponds to the availability of funds, which have been generated by other economic activities, to maintain the fishermen and the fisheries when

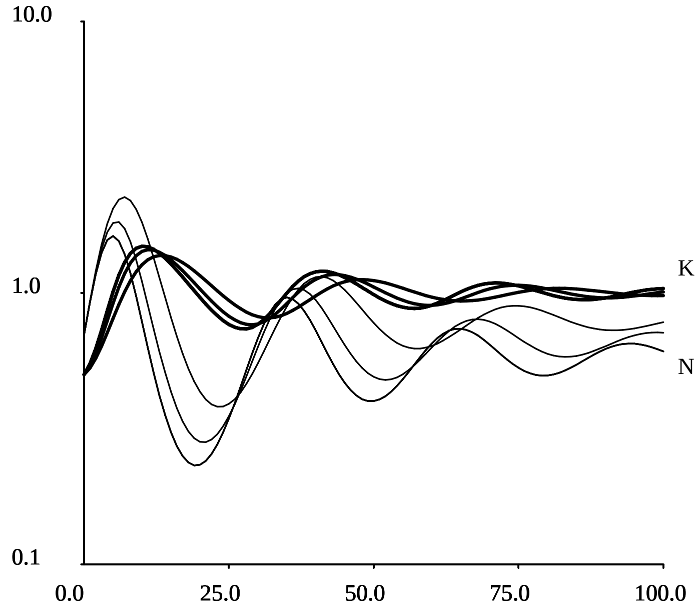


Figure 9: Time plot of the resource population and capital for the model of section 4, which takes into account the role of prices. Prices damp the oscillations, but the effect is lessened by the existence of a minimum price. The parameters of equation 37 are b=2 and c=0.5, corresponding to a maximum price of 4, and a minimum price of 0.1 (for the most damped oscillation), 1 and 2 (for the biggest oscillations).

their profits are small. On the other hand, one can be interested in the extreme case when no savings and no external income are available, and when a minimal level of consumption is necessary for the fishermen. Such would be the case, for instance, in a primitive society whose economy depends on a single resource. In order to study this case we assume that the consumption, $C$, is proportional to the profits when they are high, but that some minimum positive consumption, $c_m$, is maintained even when profits are small or negative. A simple way of expressing this is to say that the consumption is a function of $x = P_k - \mu K$, with

$$C(x) = \begin{cases} a_c x, & x > c_m \\ c_m, & x \leq c_m \end{cases} . \tag{38}$$

However, this assumption gives a consumption function which has a discontinuous derivative: besides making the mathematical analysis at $x = c_m$ difficult, this sharp transition is not realistic. A smoothed version of this function is

$$C(x) = c_m + a_c \frac{x - c_m + \sqrt{(x - c_m)^2 + c_1}}{2} \tag{39}$$

where $c_1$ is a positive constant which determines the sharpness of the corner near $x = c_m$. We assume, as before, that $a_c < 1$, so that the fraction consumed at high profits is less than 1. This consumption function, shown in figure 10, has the desired properties that $C(x) \to c_m$ as $x \to -\infty$ and $C(x) \to a_c x$ as $x \to \infty$. Both $C$ and $C'(x)$ are monotonically increasing in $x$.

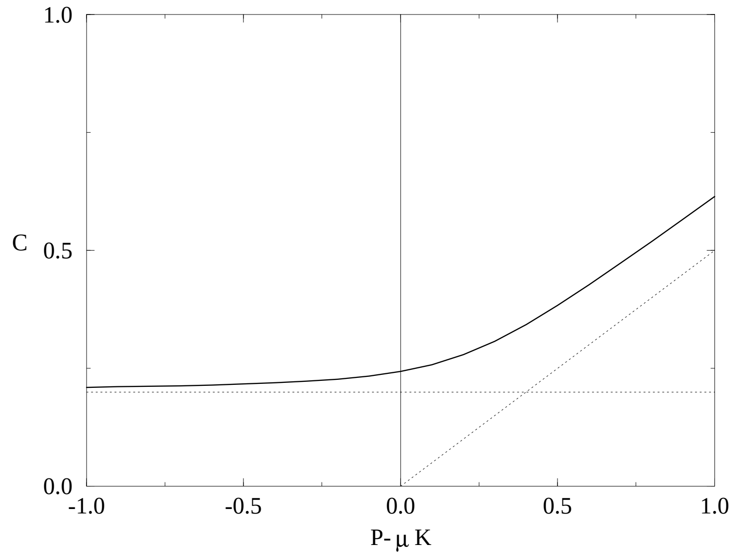


Figure 10: Consumption function used in section 5. The horizontal axis represents the profit $x = P_k - \mu K$. A minimum consumption, $c_m = 0.2$, is maintained, even in the absence of profit. When profits are are high, consumption increases in proportion to $x$, with a constant of proportionality $a_c = 0.5$ as in section 2. Here the sharpness of the corner is defined by $c_1 = 0.1$. For comparison the two asymptotes, $a_c x$ and $c_m$ are also shown in dotted lines.

The minimum consumption model is based on the set of equations

$$\frac{dN}{dt} = rN - a_n P \tag{40}$$

$$\frac{dK}{dt} = P_k - \mu K - C(P_k - \mu K) \tag{41}$$

This system is identical to that of section 2, except for the consumption function term in the capital equation.

## 5.1 Algebraic analysis

Equilibria:

There is only a single equilibrium for this new system. The $(0, 0)$ equilibrium, which existed for all previous models, is no longer a possible equilibrium state since some level of consumption still occurs even when there are no fish and no boats.

Normalizing $N$ and $K$ with respect to the $N^*$ and $K^*$ of section 2, the equilibrium with the new consumption function is at $(\tilde{K}, \tilde{N}) = (1, N_c)$, where

$$N_c = 1 + \frac{1}{\mu K^*}(c_m + \frac{a_c}{2}\sqrt{\frac{c_1}{1-a_c}}). \tag{42}$$

The positive equilibrium is at the same capital level as before, but the equilibrium number of fish is larger than the baseline value of $N^*$. Clearly this occurs because more of the capital is drained off into consumption and less can be used for harvesting fish. The equilibrium level of the resource, $N_c$, increases with the minimum consumption, $c_m$, the width of the transition region (as governed by $c_1$), and the fraction of profits are that are consumed, $a_c$.

Substituting the equilibrium values of the resource and the capital back into the expression for $C(x)$ shows that the consumption at the equilibrium point is equal to $P_k - \mu K$, or $c_m + (a_c/2)\sqrt{c_1/(1-a_c)}$. Thus when the corner in the consumption function becomes sharp, $(c_1 \to 0)$, the equilibrium is at the corner, with $x = c_m$.

Stability:

When the equilibrium point lies near the region of minimum consumption, it is strongly modified by the existence of this minimum consumption. We can still have a stable equilibrium with oscillations, but the size of the parameter region where oscillations do occur is different, and for some parameter values the equilibrium is no longer stable.

In addition, a positive minimum consumption implies that the derivative of $K$ is negative at $K = 0$: when the initial capital is small, the system might collapse even though the equilibrium is attractive (see figure 11b).

Linearizing the system around the equilibrium gives the eigenvalues

$$\lambda = -\beta_c(1 - N_c/2) \pm \sqrt{\beta_c^2(1 - N_c/2)^2 - \beta_c N_c}, \tag{43}$$

where

$$\beta_c = \frac{\beta}{2 - a_c}. \tag{44}$$

The equilibrium is thus stable as long as $N_c < 2$, i.e. when equilibrium resource level is lower than twice harvesting equilibrium. For very small values of $c_1$ this condition is equivalent to $c_m < \mu K^*$. When $c_1$ cannot be neglected the transition occurs for smaller values of $c_m$. As $c_m$ increases, the real part of $\lambda$ goes to zero, and the strength of attraction of the critical point gets gradually weaker. However, even before $N_c$ reaches 2, an unstable region can occur, and trajectories can hit the boundary if the capital gets too small (figure 11b). In fact the basin of attraction of equilibrium shrinks in the phase space as $c_m$ increases.

## 5.2 Simulation results

Once $N_c$ is greater than 2, the equilibrium becomes unstable. The analytical global stability analysis has not been done, but simulation results show that the system sometimes has a Hopf bifurcation towards a limit cycle, and sometimes simply diverges.

Oscillations occur in the vicinity of this equilibrium point when $\beta < 4N_c(2-a_c)/(2-N_c)^2$. If we are in the situation where the minimum consumption $c_m$ and the sharpness coefficient $c_1$ are small then $N_c$ is approximately 1 and the condition for oscillations simplifies (approximately) to $\beta_c < 4$, or $\beta < 4(2-a_c)$. Thus this condition does not reduce to that of the baseline model of section 2, and we should not expect it to since the equations near the equilibrium point are different.

According to whether the initial conditions are within the basin of attraction of the equilibrium or not, the addition of a minimum consumption has different effects on resource depletion (Figure 11). Inside the basin of attraction it reduces the amount of capital that can be accumulated. The resource cannot be depleted down to the levels reached without the minimum consumption assumption. This decreases the amplitude of the swings, as seen in Figures 11a and 14. But, outside the attraction basin, the lack of capital for re-investment during those hard times when consumption is maintained at its minimum results in a fast capital decrease that eventually finishes with a system crash (see figure 11b). A phase portrait of the dynamics is presented for similar conditions in figure 12.

Numerical simulations allow us to investigate the global stability. At large values of $a_c$ and $c_1$, *e.g.* .8 and .2, there exists a separatrix which prevents interior trajectories from going to infinity. When $c_m$ approaches $\mu K^*$, the inside trajectories evolve towards a limit cycle (figure 13). For smaller values of $a_c$ and $c_1$, *e.g.* .5 and .001, the separatrix shrinks when $c_m$ increases towards $\mu K^*$, and local and global stability are lost at the transition.

In any case, when the minimum consumption gets close to the capital depreciation rate at equilibrium, then the likelihood of starting from initial conditions which lead to depletion of the capital is fairly high.

# 6 Conclusions

A comparison of resource depletion for the four models is presented in figure 14.

The main emphasis of the present paper has been to interpret resource exhaustion as a dynamical depletion following an exploitation overshoot due to capital (and labor) inertia. We have seen that this qualitative result does not seem to depend upon the degree of simplification of the model since introduction of a carrying capacity, price dynamics and minimal consumption do not destroy the observed depletion. It is only when the resource renewal rate is higher than the capital depreciation that the dynamics is qualitatively changed to a monotonic decay towards equilibrium. Other refinements can still be introduced, such as endowing agents with more rational behavior, but we suspect that as long as perfect rationality is not assumed, oscillations and subsequent depletion are generic dynamical properties of the dynamics of the ecological-economical system, i.e. that they do not depend upon the details of the model. We will summarize here those results which should be robust against new refinements of the models, and which should probably be true also for a number of real systems. Of course this quest for generic properties still has to be checked by more modeling and by comparison with data from observations.

We noted in section 2 that other production functions can be considered and that, in some cases, the equilibrium could become unstable. But even in those cases the fundamental result of this study is still valid: exploitation of renewable resources with open access yields significant depletion following overshoot when resource replenishment is fast with respect to capital depreciation.

Previous dynamical models were based on fast readjustment of effort in proportion to the difference between cost and benefit [7, 16]. Our model takes into account the fact that readjustment by capital variations has a lot of inertia, which is reflected by the capital depreciation rate. Of course this inertia is responsible for the overshoots and depletions. The first type of fast readjustment of effort would correspond to some kind of perfect economic rationality, including perfect information, but human organizations are more often driven by inertia than by perfect rationality, which explains our choice for the dynamics of the economic variable. Let us note here that a limited rationality in capital investment which would avoid reallocation of capital when profit is small would not prevent resource depletion because the most profitable investment is being done when the resource is above its equilibrium value. When investors would realize that they are loosing money, capital inertia would prevent fast enough readjustment of the effort: a readjustment rate fast with respect to oscillation frequency, which is of order $\sqrt{\mu r}$ (eq. 13), would be needed instead of $\mu$ the current adjustment rate of capital. The same qualitative argument applies to labor readjustments: they could prevent resource depletion if they would occur faster than $\frac{1}{\sqrt{\mu r}}$, which could only be expected if some other sector of the economy were growing.

We will also discuss in this section some assessment of various government (or other institutional) intervention tools.

One of the simple results obtained in section 2 is that the oscillatory character of the dynamics is not changed by the production coefficients $a_k$ and $a_n$. In other words, technological and marketing improvement won't save us from overshoots and depletion, but can increase the level of the renewable resource at equilibrium.

We have seen that decreasing the capital depreciation rate, $\mu$, and thus $\lambda$, amplifies the oscillations and level of resource depletion. In the case where government intervention consists of supporting investment by providing loans with low interest rates, $\lambda$ is probably decreased and the instability increased. This is one example of a situation where government support of the industry, or of the fishermen, could be detrimental to the fish stock, and thus to the industry in the long term [3].

We have seen in section 3 that the oscillations and resource depletion are strongly attenuated when the fishing effort is such that the harvesting equilibrium is in the neighborhood of the carrying capacity . But, of course, the effect is only important when the fishing effort is small. If the harvesting equilibrium is far below the carrying capacity, oscillation damping is quite reduced.

Market mechanisms and price variations can reduce resource depletion, but we have seen that this reduction is limited to a factor of two in the real part of the time constant, see eqn. 35. This prediction is made for a rather simple model, however our assumption that the price decreases inversely proportional to the production is probably an extreme since it corresponds to an income that saturates at high yields. We expect instead that prices drop less rapidly in normal situations, and thus that the reduction in resource depletion should be less important than the one predicted in section 4. The reduction in resource depletion by the market only occurs when the demand is sufficient to avoid an external support of prices at production. In this context, institutional support prevents the reduction of resource depletion by market mechanisms and might reduce the ecological sustainability of the system in the absence of any complementary measure.

Societies have introduced financial mechanisms which allow them to maintain the personal income of producers even when natural production is plummeting. The effect of maintaining a minimum consumption level has been investigated in section 5. Once more the dynamics shows that the system sustainability is decreased when a minimum consumption is maintained.

The above results converge in pointing out the fact that external intervention has to balance the short term interests of labour and capital with their long term interests when the renewable resource is in danger of being depleted.

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# References


[1] P. M. Allen and J. M. McGlade. 1986. Modelling complexity: the dynamics of discovery and exploitation in a fisheries example. *Mondes en développement*, **14**:49–76.

[2] F. Bousquet, C. Cambier, C. Mullon, P. Morand, J. Quensiere, and A. Pave. 1993. Simulating the interaction between a society and a renewable resource. *Journal of biological systems*, **1**:199–214.

[3] CEE. 1991. Rapport 1991 de la commission au conseil et au parlement sur la politique commune de la pêche. Technical Report SEC(91) 2288, CEE.

[4] A. T. Charles. 1983. Optimal fisheries investment: comparative dynamics for a deterministic seasonal fishery. *Canadian journal of fisheries and aquatic sciences*, **40**:2070–2079.

[5] A. T. Charles. 1988. Fishery socioeconomic: A survey. *Land Economics*, **64**:276–295.

[6] A. T. Charles. 1989. Bio-socio-economic fishery models: labour dynamics and multiobjective management. *Canadian journal of fisheries and aquatic sciences*, **46**:1313–1322.

[7] C. W. Clark. 1990. *Mathematical bioeconomics: The optimal management of renewable resources (2nd edition)*. Wiley-Intersciences, New-York.

[8] Cropper, Lee, and Pannu. 1979. The optimal extinction of a renewable natural resource. *Journal of International Economics*, **6**:341–349.

[9] R. DeBoer. 1983. Grind: Great integrator for differential equations. Technical report, BioInformatics Group, University of Utrecht.

[10] J. R. Durand, J. Lemoalle, and J. Weber. 1989. *La recherche face à la pêche artisanale*. Orstom éditions, Montpellier.

[11] H. S. Gordon. 1954. The economic theory of a common property resource: the fishery. *Journal of political economy*, **62**:124–142.

[12] P. A. Larkin. 1977. An epitaph for the concept of maximum sustained yield. *Transactions of the american fisheries society*, **106**:1–11.

[13] R. M. May. 1981. *Theoretical ecology: principles and applications*. Blackwell Scientific Publications, Oxford, page 489.

[14] D.H. Meadows, D.L. Meadows, and J. Randers. 1972. *The Limits to growth*. Universe books, New York.

[15] D.H. Meadows, D.L. Meadows, and J. Randers. 1992. *Beyond the Limits: Confronting Global Analysis, Envisioning a Sustainable Future.* Chelsea Green Publishing Company, Post Mills, Vt.

[16] J.D. Opsomer and J. M. Conrad. 1994. An open-access analysis of northern anchovy fishery. *Journal of environmental economics and management*, **27**:21–37.

[17] J. Roughgarden and G. Brown. 1993, An ecological economy. Unpublished.

[18] J. Samuelson. 1993. *Economy.* McGraw-Hill, New York.

[19] M. B. Schaefer. 1954. Some aspects of the dynamics of populations important to the management of commercial marine fisheries. *Bulletin of the Inter-American tropical tuna commission*, **1**:25–56.

[20] V. L. Smith. 1968. Economics of production from natural resources. *Amer. Econom. Rev.*, **58**:409–431.

[21] J.E. Stiglitz. 1979. A neoclassical analysis of the economics of natural resources. In Smith and Kerry, editors, *Scarcity and Growth Reconsidered*, pages 36–66. Johns Hopkins, Baltimore.

[22] S. S. Yoshimoto and R. P. Clarke. 1993. Comparing dynamic versions of the Shaefer and Fox production models and their application to lobster fisheries. *Canadian Journal of fisheries and aquatic sciences*, **50**:181–189.

# 7 Appendix: Analysis of Variable Prices

Here we present the analysis for the case described in Section 4, of prices with a lower bound.

Suppose that prices time $a_n$ are given by equation 37 with $b/c > a > 0$, so that prices decrease from a maximum of $b/(a_n c)$ toward a minimum of $a/a_n$. In order to make comparisons with the results of the other sections, a reasonable assumption is that the minimum price is lower than under the constant price regime of section 2, i.e. that $a < a_k < b/c$, where $a_k/a_n$ is the price of section 2. Using equation 37 implies that there is no limit on the income from selling the harvest, as there is when $a = 0$. However,as long as $a$ is small compared to 1, we expect that the results given for the $a = 0$ case will essentially hold: namely that a nonzero equilibrium only exists when $b$ is above a minimum value near $\mu K^*$, and that (when $b > \mu K^*$) the equilibrium resource level is larger, the equilibrium is more strongly attracting, and oscillations are less likely than in the constant price case. However, as $a$ moves away from zero and price supports become stronger, we would like to know how this affects the $a = 0$ results.

When $a > 0$ the nonzero equilibrium of the nondimensionalized system with $\tilde{N} = N/N^*$, $\tilde{K} = K/K^*$ ($N^*$, $K^*$ given by equation 5) is $\tilde{K} = K_p$, $\tilde{N} = N_p$, where

$$K_p = 1, \quad N_p = \frac{1 - \tilde{b} + \sqrt{(1-\tilde{b})^2 + 4\tilde{a}\tilde{c}}}{2\tilde{a}}; \tag{45}$$

and we define $\tilde{a} = a/a_k$, $\tilde{b} = b/(\mu K^*)$ $\tilde{c} = c/P^*$, and $P^* = \sqrt{LK^*}N^* = \mu K^*/a_k$.

As we found in section 4, adding prices does not change the equilibrium capital but significantly affects the equilibrium resource. However, unlike the $a = 0$ case, $N - p$ is positive for all positive values of $b$. What is different now is that if $b < 1$ then $N_p \to \infty$ as $a \to 0$ while if $b > 1$ then $N_p$ approaches the $a = 0$ value as $a \to 0$.

Letting

$$\alpha = \frac{\tilde{b}\tilde{c} + 2\tilde{a}\tilde{c}N_p + \tilde{a}(N_p)^2}{(\tilde{c} + N_p)^2}, \tag{46}$$

and linearizing about the nontrivial equilibrium, we find that the eigenvalues of this system are:

$$2\lambda = -\beta(1 - \alpha N_p/2) \pm \sqrt{\beta^2(1 - \alpha N_p/2)^2 - 2\beta\alpha N_p} \tag{47}$$

These eigenvalues always have negative real part. This can be seen via the following argument. Note that in the remaining discussion of this section, we drop the tildes. Since $\alpha > 0$, these eigenvalues both have negative real part when $2 \geq \alpha N_p$. Using the relationship between $a$, $b$, $c$ and $N_p$ to replace each $N_p^2$, this condition on $\alpha N_p$ can be rewritten as

$$2(c + N_p)^2 \geq N_p(aN_p^2 + 2acN_p + bc) = (c + bc)N_p + (1 - b + 2ac)N_p^2, \tag{48}$$

or

$$2c^2 + (3c - bc)N_p + (1 + b - 2ac)N_p^2 \geq 0, \tag{49}$$

and applying the equality again shows that this is always true. Thus this is always a stable equilibrium point. It is a stable spiral point when

$$\beta < \frac{8\alpha N_p}{(2 - \alpha N_p)^2}. \tag{50}$$

In the limit as $a \to 0$, these results approach those found in section 3. We expect that increasing $a$ will have the effect of annuling the impact of decreasing prices, and, in fact, increasing the lower bound on prices, $a/a_n$, decreases the equilibrium level of the resource, $N_pN^*$. However, the dependence of the eigenvalues on $a$ is much more subtle, and depends on the values of $b$ and $c$. For very small $a$, the upper bound on $\beta$ for oscillations decreases as $a$ increases (when $\tilde{b}$

satisfies the constraint $\tilde{b} > 1$), implying a stabilizing effect. When the parameters are such that $\beta$ satisfies this inequality and the value under the square root is negative, then the real part of $\lambda$ increases in magnitude with $a$ when $1 < b < 3$, and otherwise decreases.

These results are found by the following calculations:

$$\frac{\partial N_p}{\partial a} = \frac{c}{a\sqrt{(1-b)^2 + 4ac}} - N_p/a, \tag{51}$$

which is negative for all $a > 0$.

$$\lim_{a \to 0} \frac{\partial(\beta(1 - \alpha N_p/2))}{\partial a} = \frac{\beta c(3-b)}{2(b-1)} \tag{52}$$

and

$$\lim_{a \to 0} \frac{\partial}{\partial a}\left(\frac{8\alpha N_p}{(2 - \alpha N_p)^2}\right) = -\frac{8(3-b)^2 c}{(b-1)(b+1)^3}. \tag{53}$$

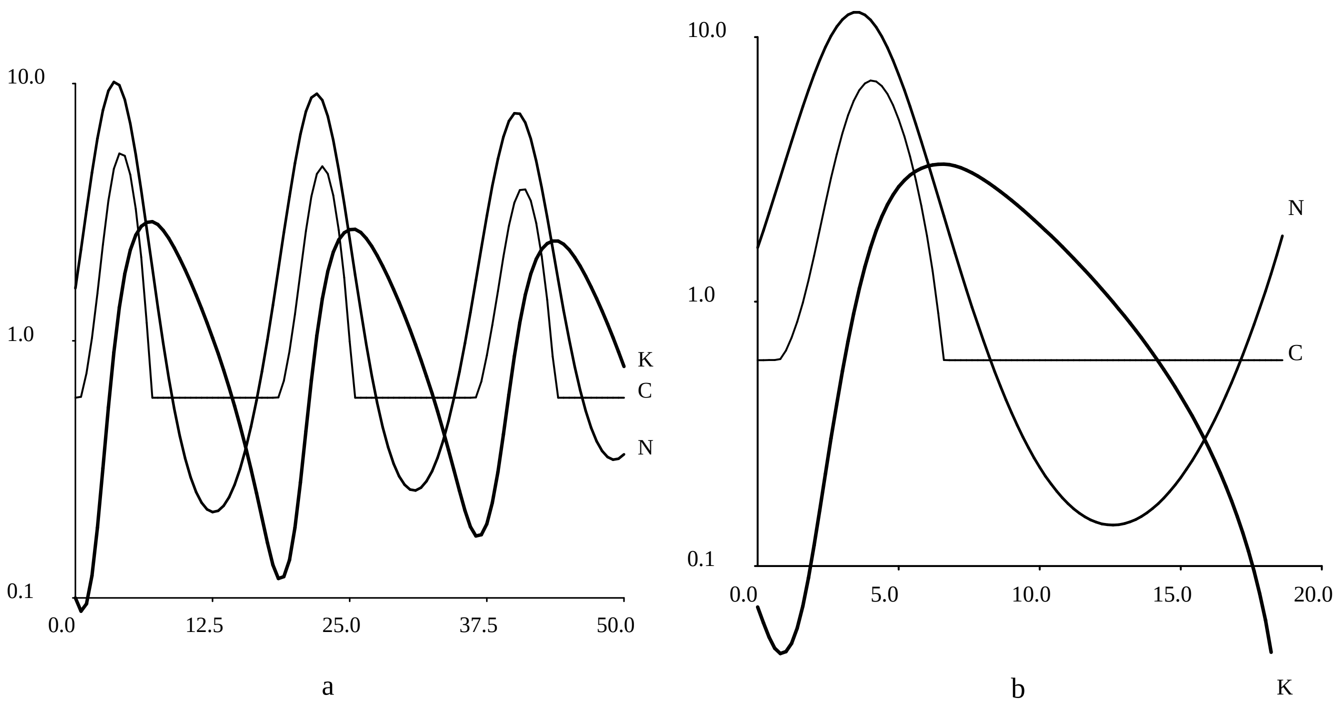


Figure 11: Time plots of the logarithms of resource, capital, and consumption for the model of section 5. Here $N$ and $K$ are scaled by $N^*$ and $K^*$, respectively, of section 2, and the consumption $C$ is scaled by $\mu K^*$. Shown are two different initial conditions for the case $\beta = 0.1$, $a_c = 0.5$, $c_m = 0.6\mu K^*$, and $c_1 = 0.001(\mu K^*)^2$. In both cases $N$ starts out at its equilibrium value of $1.6N^*$. Each time the profit plummets to $c_m$ or less, the consumption bottoms out to $c_m$. (a) $K(0) = 0.1K^*$. In this case the initial conditions are in the attracting region of the equilibrium, and capital never goes to zero. (b) $K(0) = 0.07$. Although this initial condition is near to that of (a), it is just outside the zone of attraction of the equilibrium. The amplitude of the oscillations gradually grows, until capital collapses.

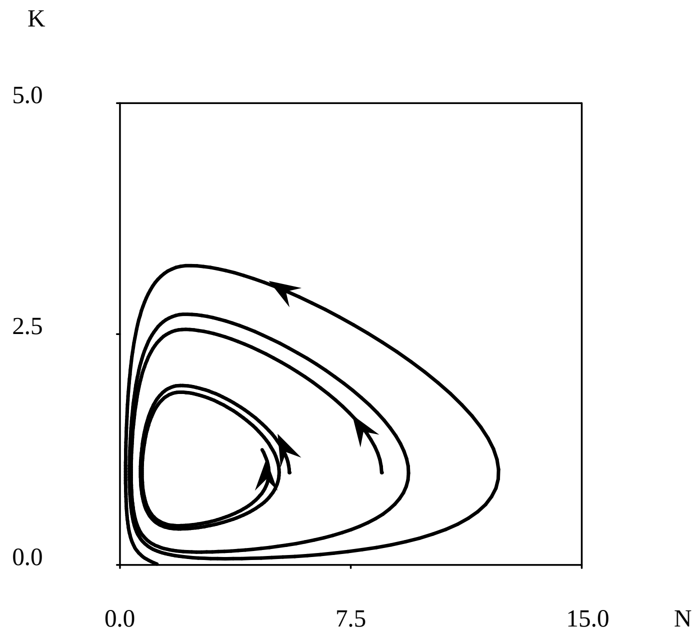


Figure 12: The unstable limit cycle (separatrix) of the minimum consumption model of section 5 shown on the phase portrait. The equilibrium point is surrounded by an unstable limit cycle. Inside the separatrix solutions approach the equilibrium, while outside oscillations increase in amplitude until the boundary $K = 0$ is reached. This plot shows two trajectories for the case $\beta = 0.1$, $a_c = 0.5$ $c_m/(\mu K^*)$ =0.8; $c_1/(\mu K^*)^2 = 0.001$. $K/K^*$ is plotted against $N/N^*$. Arrows show the direction of the trajectory as time increases. Initial coordinates are $(\tilde{N}, \tilde{K}) = (5.5, 1)$ and $(8.5, 1)$.

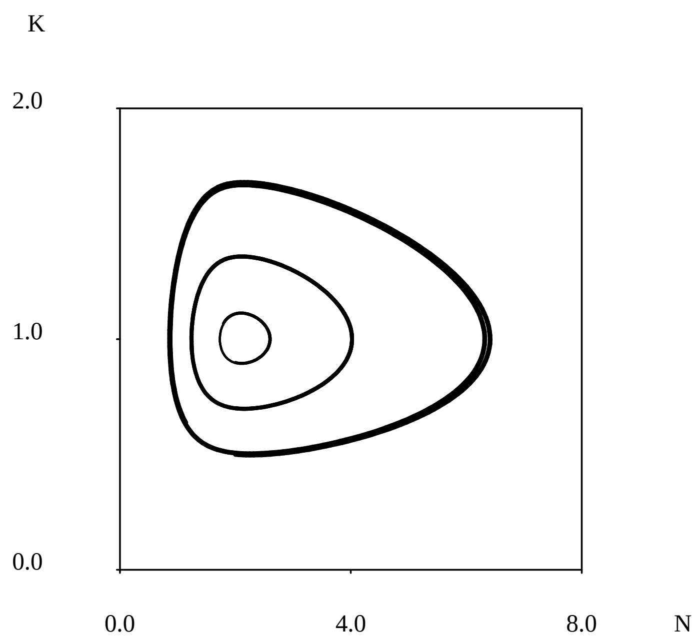


Figure 13: A stable limit cycle of the minimum consumption model of section 5. A limit cycle is observed above the equilibrium instability transition obtained at large minimum consumption. This limit cycle is shown for the case $a_c = 0.8$, $c_m/\mu K^* = 0.7$, $c_1/(\mu K^*)^2 = 0.2$, $r/\mu = 0.2$, for 3 different sets of initial conditions. It is difficult to see, but the inside trajectory ($K(0) = 0.9K^*$) is gradually moving outward, the outside trajectory ($K(0) = 0.5K^*$) moves inward, and the center trajectory $K(0) = 0.7K^*$, is essentially periodic. In each case, $N$ starts at the equilibrium value of $2.1N^*$. The region of attraction of the limit cycle is limited: not shown is an unstable limit cycle separating the region of attraction of the inner limit cycle from an unstable region where solutions always reach the $K = 0$ boundary, just as in the previous figure.

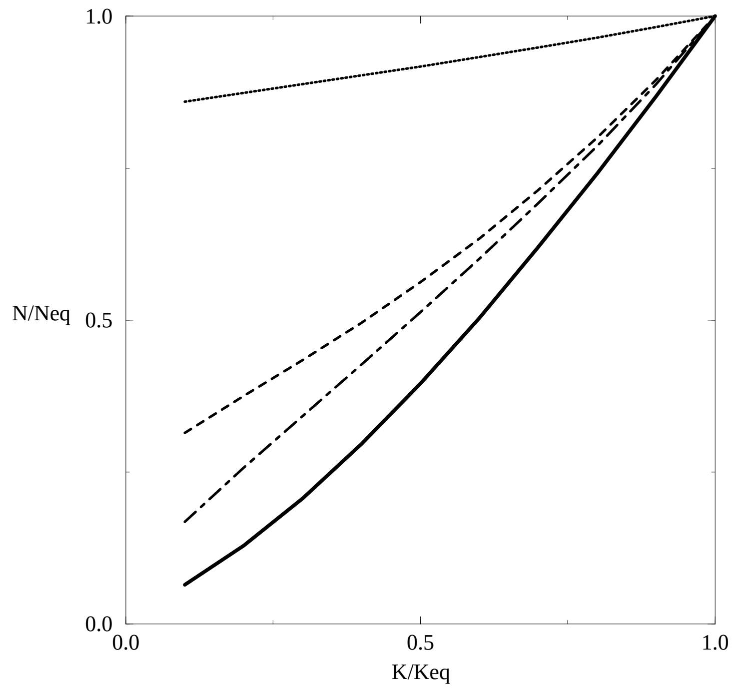


Figure 14: Model comparison: Magnitude of the resource depletion as a function of the distance from equilibrium for each of the four models studied in this paper. The fraction of initial capital with respect to equilibrium capital is plotted along the horizontal axis, and the fraction of resource at maximum depletion with respect to equilibrium resource along the vertical axis. Solid line: the basic model of section 2 with $\beta = 0.1$, $\tilde{N}_{eq} = \tilde{K}_{eq} = 1$; Dots: the model of section 3 with a carrying capacity ($\beta = 0.1, \tilde{m} = 5$, $\tilde{N}_{eq} = 0.83$, $\tilde{K}_{eq} = 0.69$); Dashes: the price model section 4 ($\beta = 0.1, a = 0.1$, $b = 2.0$, $c = 0.5$, $\tilde{N}_{eq} = 0.4772$, and $\tilde{K}_{eq} = 1.0$); Dash dot: the minimum consumption model section 5 ($\beta = 0.1$, $a_c = 0.5$, $c_m = 0.1\mu K^*$, $c_1 = 0.05(\mu K^*)^2$, $\tilde{N}_{eq} = 1.27906$, and $\tilde{K}_{eq} = 1$).